Discrete Mathematics - First Midterm Exam Warm-up

1. Assume that in a group of six people, each pair of individuals consists of two friends or two enemies. Show that in this group either there are three mutual friends or three mutual enemies.

2. Show that among any $n+1$ positive integers not exceeding $2n$, there must be two sum of elements of which is divisible by n, and among any $n+2$ positive integers not exceeding $2n$, there must be two sum of elements of which is divisible by $2n$.

3. Prove (using induction) that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

4. How many ways are there to put fruits in a basket, so that the basket is not empty, if we have n indistinguishable apples and m indistinguishable oranges.

5. What is the number of bit strings of length 100 with exactly 20 ones?

6. How many permutations of the letters of the word ENCYCLOPEDIA contain a substring EE **and** a substring CC?

7. How many permutations of the letters of the word ENCYCLOPEDIA contain a substring EE **or** a substring CC?

8. Each user on a computer system has a password, which is ten characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

9. Find the number of positive integers not exceeding 100 that are not divisible by 4 nor by 6 nor by 15.

10. How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 20$, $x_1 \geq 1$, $x_2 \geq 2$, $x_3, x_4 \geq 0$?

11. How many ways are there to distribute ten different sandwiches to three different children such that each child gets at least one sandwich?

12. Let a_n be the number of subsets of the set $[n] = \{1, \ldots, n\}$, which do not have pairs of the type $k, k+1$. Find a recurrence formula for a_n.

13. In how many ways can a $2 \times n$ rectangular checkerboard be tiled using 1×2 and 3×2 pieces? Find a suitable recurrence formula.

14. Solve the following recurrence relations

$$a_n = a_{n-1} + 12a_{n-2}, n \geq 2,\ a_0 = 0, a_1 = 2,$$

$$b_n = b_{n-1} + n^2, n \geq 1,\ b_0 = 2.$$

Hints and solutions

1. Use the Pigeonhole Principle.

2. Use the Pigeonhole Principle with partition $\{1, 2n-1\}, \{2, 2n-2\}, \ldots, \{n, 2n\}$.

3. Use general induction in the form $S(n-5), S(n-4) \Rightarrow S(n)$.

As a base step prove $S(12), S(13), S(14), S(15)$.

4. $(n+1)(m+1) - 1 = nm + n + m$

5. $\binom{100}{20}$

6. $10!$

7. $11! - 10!$ (use the inclusion-exclusion principle)

8. $36^{10} - 26^{10}$

9. 64 (Count first the number of integers that are divisible by 4, by 6 or by 15.

To this end use the inclusion-exclusion principle).

10. $\binom{20}{17}$

11. $3^{10} - 3 \cdot 2^{10} + 3$ (use the inclusion-exclusion principle)

12. $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$, $a_1 = 1$, $a_2 = 2$

13. $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, $n \geq 4$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$

14. $a_n = \frac{2}{7} \cdot 4^n - \frac{2}{7} \cdot (-3)^n$, $b_n = 2 + \frac{n(n+1)(2n+1)}{6}$